ESTIMATES FOR THE NORM OF THE DERIVATIVE OF LIE EXPONENTIAL MAP FOR CONNECTED LIE GROUPS

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Abstract.

Let G be a connected real Lie group with a left invariant Riemannian metric d, and $\mathfrak{g}$ be its Lie Algebra as an inner product space, and $\exp : \mathfrak{g} \to G$ the Lie exponential map. For $g \in G$ let l_g denote the left multiplication by g. One important question that arise about the exponential map would be asking if there are conditions under which the exponential map is quasi-isometry. Quasi-isometries are mappings between two metric spaces, or in the context of present topic, between two Riemannian manifolds that respect large-scale geometry of these spaces and ignore their small-scale details.

This is true if the universal covering of G is $\mathbb{R}^n$. The other conditions that might be worthy of investigation are when G is compact, semi-simple, solvable or nilpotent.

Answering the 'quasi-isometry' question raises the problem of bounding the image of the differential of the exponential map. More specifically, given a non-zero vector $x \in \mathfrak{g}$ and any vector $y \in \mathfrak{g}$, we would like to find upper and lower bounds for $|d\exp_x(y)|$. It is well known that the differential of the exponential map at x is given by

$$d\exp_x = dl_{\exp(x)} \frac{1 - e^{-\mathrm{ad}_x}}{\mathrm{ad}_x} .$$

Since the metric d is left invariant, it follows that for any vector $y \in \mathfrak{g}$

$$|d\exp_x(y)| = \left| \frac{1 - e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}(y) \right| \tag{1}$$

Thus the problem of finding upper and lower bounds for $|d\exp_x(y)|$ would be equivalent to finding estimates for the norm of the image of

$$\frac{1 - e^{-\mathrm{ad}_x}}{\mathrm{ad}_x} ,$$

which can be regarded as a compact operator on $\mathfrak{g}$ as a finite dimensional Hilbert space. Singular values of a linear operator defined on a finite dimensional Hilbert space are related to the maximum of minimums (and the minimum of the maximums) of the norm of the operator on some subspaces of $\mathcal{H}$. This is known as minimax principle for singular values.

In the previous presentation, I used this principle to bound the norm of the image of the exponential map only when ad_x is diagonalizable as stated in the following Theorem:

Theorem 1. *Let* $x \in \mathfrak{g}$ *be non-zero. Let* $\hat{x} = x/|x|$, $\lambda_1, \cdots, \lambda_p \in \mathbb{C}$ *be non-zero eigenvalues of* $\mathrm{ad}_{\hat{x}}$, *and*

$$\tilde{\lambda}_{\min}(|x|) = \min\left\{1, \left|\frac{1-e^{-\lambda_1|x|}}{\lambda_1|x|}\right|, \cdots, \left|\frac{1-e^{-\lambda_p|x|}}{\lambda_p|x|}\right|\right\},$$
$$\tilde{\lambda}_{\max}(|x|) = \max\left\{1, \left|\frac{1-e^{-\lambda_1|x|}}{\lambda_1|x|}\right|, \cdots, \left|\frac{1-e^{-\lambda_p|x|}}{\lambda_p|x|}\right|\right\}.$$

Then there exist positive constants C, D, *only depending on* $\hat{x}$, *such that for any unit vector* $y \in \mathfrak{g}$

$$C\tilde{\lambda}_{\min}(|x|) \leq |d\exp_x(y)| \leq D\tilde{\lambda}_{\max}(|x|)\,. \tag{0.1}$$

In my new results I use the Jordan canonical form, together with identities involving analytic images of Jordan blocks, and a lower bound for the smallest singular value (paper by Y. P. Hong and C.-T. Pan) to generalize Theorem 1 to all non-zero $x \in \mathfrak{g}$ including when ad_x is not diagonalizable.